

# An Algebraic Approach to Shadowed Sets

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## Abstract

After the introduction of shadowed sets and the investigation of their relation with fuzzy sets, we present  $BZMV^{dM}$  algebras as an abstract environment for both shadowed and fuzzy sets. Then, we introduce the weaker notion of pre- $BZMV^{dM}$  algebra. This structure enables us to algebraically define a mapping from fuzzy sets to shadowed sets.

*Key words:* Shadowed sets, fuzzy sets, BZMV algebras, rough approximations.

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## 1 Fuzzy and Shadowed Sets

In this section, we introduce the basic notions of fuzzy and shadowed sets, and outline the relation existing between them. First, we give the definition of fuzzy sets. The reader interested in the, widely studied, fields of fuzzy sets and fuzzy logic, can refer to some classical text (see, for example, [5,10]).

**Definition 1.1** Let  $X$  be a set of objects, called the Universe. A *fuzzy set* on  $X$  is any mapping  $f : X \rightarrow [0, 1]$ . We denote the collection of all fuzzy sets on  $X$  as  $[0, 1]^X$  or sometimes simply by  $\mathcal{F}$ .

The role of a fuzzy set is to describe vagueness: given a vague concept  $f$  on a universe  $X$ , the value  $f(x)$  indicates the degree to which  $x$  belongs to the concept  $f$ . One feature of such an approach is the description of a vague concept through an exact numerical quantity. A different approach to vagueness has been proposed by Pedrycz ([7,8,9]). His intention was “to introduce a model which does not lend itself to precise numerical membership values but relies on basic concepts of truth values (yes - no) and on entire

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unit interval perceived as a zone of uncertainty” ([7]). This idea of modeling vagueness through vague (i.e., not purely numeric) information, lead him to the definition of shadowed sets.

**Definition 1.2** Let  $X$  be a set of objects, called the Universe. A *shadowed set* on  $X$  is any mapping  $s : X \rightarrow \{0, 1, (0, 1)\}$ . We denote the collection of all shadowed sets on  $X$  as  $\{0, 1, (0, 1)\}^X$ .

In the sequel we will indicate  $(0, 1)$  with the value  $\frac{1}{2}$ . This will simplify our algebraic approach from a syntactical point of view, without losing the semantic of “total uncertainty” of the value  $(0, 1)$ . In fact, if 1 corresponds to truth, 0 to falseness, then  $\frac{1}{2}$  is halfway between true and false, i.e., it represents a really uncertain situation.

From a fuzzy set it is possible to obtain a shadowed set. Let  $f$  be a fuzzy set; then, it is sufficient to define a value  $\alpha \in [0, \frac{1}{2})$  and set to 0 the membership values  $f(x)$  which are less than or equal to  $\alpha$  and set to 1 those greater than  $(1 - \alpha)$ . The membership values belonging to  $(\alpha, 1 - \alpha)$  are those characterized by a great uncertainty or lack of knowledge and they are consequently considered the “shadow” of the induced shadowed set, i.e., they are set to  $\frac{1}{2}$ .

In a more formal way, once fixed a value  $\alpha$ , we can define the  $\alpha$ -approximation function of a fuzzy set  $f$ , denoted by  $s_\alpha(f)$ , as the following shadowed set:

$$s_\alpha(f)(x) := \begin{cases} 0 & f(x) \leq \alpha \\ 1 & f(x) \geq 1 - \alpha \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad (1)$$

In Figure 1 it is represented a fuzzy set and the induced shadowed set.

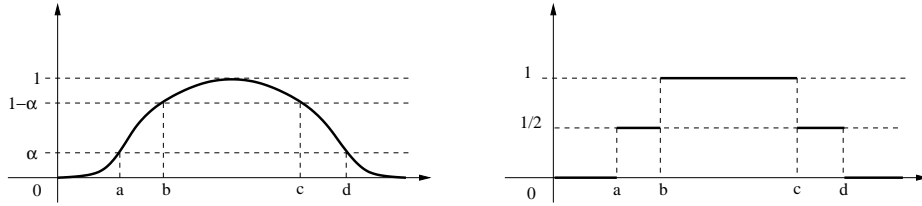


Fig. 1. A fuzzy set and its corresponding shadowed set

## 2 An algebraic framework

As an algebraic approach to both fuzzy and shadowed sets we propose  $BZMV^{dM}$  algebras ([2,3]).

**Definition 2.1** A *de Morgan Brouwer Zadeh Many Valued* ( $BZMV^{dM}$ ) algebra is a system  $\langle A, \oplus, \neg, \sim, 0 \rangle$ , where  $A$  is a non empty set,  $\oplus$  is a binary operator,  $\neg$  and  $\sim$  are unary operators, 0 is a constant, obeying the following axioms:

$$(BZMV1) \quad (a \oplus b) \oplus c = (b \oplus c) \oplus a$$

$$(BZMV2) \quad a \oplus 0 = a$$

$$(BZMV3) \quad \neg(\neg a) = a$$

$$(BZMV4) \quad \neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$$

$$(BZMV5) \quad \sim a \oplus \sim \sim a = \neg 0$$

$$(BZMV6) \quad a \oplus \sim \sim a = \sim \sim a$$

$$(BZMV7) \quad \sim \neg[(\neg(a \oplus \neg b) \oplus b)] = \neg(\sim \sim a \oplus \neg \sim \sim b) \oplus \neg \sim \sim b$$

On a  $BZMV^{dM}$  algebra, it is possible to derive the following further operators:

$$a \odot b := \neg(\neg a \oplus \neg b)$$

$$a \vee b := \neg(\neg a \oplus b) \oplus b$$

$$a \wedge b := \neg(\neg(a \oplus \neg b) \oplus \neg b)$$

Connectives  $\vee$  and  $\wedge$  are the algebraic realization of logical disjunction and conjunction of a *distributive lattice*  $\mathfrak{L}$  in particular, they are idempotent operators. Connectives  $\oplus$  and  $\odot$  are the well known MV disjunction and MV conjunction operators, which are not idempotent ([10]). A partial order can be naturally induced by the lattice operators as:

$$a \leq b \text{ iff } a \wedge b = a \quad (\text{equivalently, } a \vee b = b)$$

Let us notice that, since it is possible to prove that  $\sim 0 = \neg 0$ , in the sequel we set  $1 := \sim 0 = \neg 0$ . With respect to the just defined partial order we have that the lattice is bounded:  $\forall a \in A, 0 \leq a \leq 1$ .

The unary operation  $\neg : A \mapsto A$  is a *Kleene* (or *Zadeh*) orthocomplementation (negation). In other words, it satisfies the properties:

$$(K1) \quad \neg(\neg a) = a$$

$$(K2) \quad \neg(a \vee b) = \neg a \wedge \neg b$$

$$(K3) \quad a \wedge \neg a \leq b \vee \neg b$$

Let us recall that under (K1), condition (K2) is equivalent to the dual de Morgan law. In general neither the non-contradiction law,  $\forall a : a \wedge \neg a = 0$ , nor the excluded middle law,  $\forall a : a \vee \neg a = 1$ , are satisfied by this negation.

The unary operation  $\sim : A \mapsto A$  is a *Brouwer* orthocomplementation (negation). In other words, it satisfies the properties:

$$(B1) \quad a \wedge \sim \sim a = a \quad (\text{equivalently, } a \leq \sim \sim a)$$

$$(B2) \quad \sim(a \vee b) = \sim a \wedge \sim b$$

$$(B3) \quad a \wedge \sim a = 0$$

In general from (B1)–(B3) neither the excluded middle law  $\forall a, a \vee \sim a = 1$  nor the dual de Morgan law  $\sim(a \wedge b) = \sim a \vee \sim b$  can be deduced.

Using the above definitions, we can justify the qualification of *de Morgan* given to BZMV algebras in Definition 2.1. In fact, it can be proved that

BZMV<sup>dM</sup> algebras satisfy all de Morgan properties:

$$\begin{aligned}\neg(a \wedge b) &= \neg a \vee \neg b & \neg(a \vee b) &= \neg a \wedge \neg b \\ \sim(a \wedge b) &= \sim a \vee \sim b & \sim(a \vee b) &= \sim a \wedge \sim b\end{aligned}$$

Besides, it is possible to define, through the interaction of the two unary operations  $\neg$  and  $\sim$ , the modal operators of *necessity*  $\nu(a) := \sim \neg a$ , and *possibility*  $\mu(a) := \neg \sim a = \neg \nu(\neg a)$ .

These modal operators turn out to have an  $S_5$ -like behavior based on a Kleene algebra, instead of on a Boolean one ([4]).

**Proposition 2.2** *In any BZMV<sup>dM</sup> algebra the following conditions hold:*

- (1)  $\nu(a) \leq a \leq \mu(a)$ . *In other words: necessity implies actuality and actuality implies possibility (a characteristic principle of the modal system T).*
- (2)  $\nu(\nu(a)) = \nu(a)$ ,  $\mu(\mu(a)) = \mu(a)$ . *Necessity of necessity is equal to necessity; similarly for possibility (a characteristic  $S_4$ -principle).*
- (3)  $a \leq \nu(\mu(a))$ . *Actuality implies necessity of possibility (a characteristic B-principle).*
- (4)  $\mu(a) = \nu(\mu(a))$ ,  $\nu(a) = \mu(\nu(a))$ . *Possibility is equal to the necessity of possibility; whereas necessity is equal to the possibility of necessity (a characteristic  $S_5$ -principle).*

As a consequence of the above definitions we have that  $\sim a = \neg \nu(a)$ , that is the Brouwer complement can be interpreted as the negation of possibility or *impossibility*.

As stated in Proposition 2.2, for any element of a BZMV<sup>dM</sup> algebra the order chain  $\nu(a) \leq a \leq \mu(a)$  holds. We are, now, interested to those elements which satisfy the strongest condition  $\nu(e) = e$  (equivalently,  $e = \mu(e)$ ), i.e., to those elements which present the classical feature that actuality coincide with necessity and possibility. These elements are called *sharp* (exact, crisp) elements (in contraposition to the elements which are fuzzy) and their collection is denoted by  $A_e$ .

**Remark 2.3** This is not the only way to define sharp elements. In fact, since in general  $x \wedge \neg x \neq 0$  (equivalently,  $x \vee \neg x \neq 1$ ) it is possible to consider as *Kleene sharp* (K-sharp) the elements which satisfy the non contradiction (or equivalently the excluded middle) law with respect to the Kleene negation:  $A_{e,\neg} := \{e \in A : e \wedge \neg e = 0\} = \{e \in A : e \vee \neg e = 1\}$ . Alternatively, considering the Brouwer negation we have that, in general, the double negation law does not hold (see the (B1)). So, we can introduce a further definition of *Brouwer sharp* (B-sharp) elements:  $A_{e,B} := \{e \in A : \sim \sim e = e\}$ . Finally, as said before  $\oplus$  is not an idempotent operator. So the  $\oplus$ -sharp elements are:  $A_{e,\oplus} = \{e \in A : e \oplus e = e\}$ . However, it can be proved that all this notions are equivalent ([2,3]). Let  $A$  be a BZMV<sup>dM</sup> algebra, then  $A_e = A_{e,B} = A_{e,\oplus} = A_{e,\neg}$ . Consequently, we simply talk of *sharp* elements and write  $A_e$  to denote their

collection.

Given an element  $a$  of a  $\text{BZMV}^{dM}$  algebra, modal operators,  $\nu$  and  $\mu$ , can be used to give a rough approximation of  $a$  by sharp definable elements. In fact,  $\nu(a)$  (resp.,  $\mu(a)$ ) turns out to be the best approximation from the bottom (resp., top) of  $a$  by sharp elements. To be precise, for any element  $a \in A$  the following holds:

- (I1)  $\nu(a)$  is sharp ( $\nu(a) \in A_e$ ).
- (I2)  $\nu(a)$  is an *inner* (lower) approximation of  $a$  ( $\nu(a) \leq a$ ).
- (I3)  $\nu(a)$  is the best inner approximation of  $a$  by sharp elements (let  $e \in A_e$  be such that  $e \leq a$ , then  $e \leq \nu(a)$ ).

Analogously

- (O1)  $\mu(a)$  is sharp ( $\mu(a) \in A_e$ ).
- (O2)  $\mu(a)$  is an *outer* (upper) approximation of  $a$  ( $a \leq \mu(a)$ ).
- (O3)  $\mu(a)$  is the best outer approximation of  $a$  by sharp elements (let  $f \in A_e$  be such that  $a \leq f$ , then  $\mu(a) \leq f$ ).

**Definition 2.4** Given a  $\text{BZMV}^{dM}$  algebra  $\langle A, \oplus, \neg, \sim, 0 \rangle$ , the induced *rough approximation space* according to [1] is the structure  $\langle A, A_e, \nu, \mu \rangle$  consisting of the set  $A$  of all approximable elements, the set  $A_e$  of all *definable* (or sharp) elements, and the *inner* (resp., *outer*) *approximation map*  $\nu : A \rightarrow A_e$  (resp.,  $\mu : A \rightarrow A_e$ ).

For any element  $a \in A$ , its *rough approximation* is defined as the pair of sharp elements:  $r(a) := \langle \nu(a), \mu(a) \rangle$  [with  $\nu(a) \leq a \leq \mu(a)$ ].

So the map  $r : A \rightarrow A_e \times A_e$  approximates an unsharp (fuzzy) element by a pair of exact ones representing its inner and outer sharp approximation, respectively. Clearly, sharp elements are characterized by the property that they coincide with their rough approximations:  $e \in A_e$  iff  $r(e) = \langle e, e \rangle$ . An equivalent way to define a rough approximation space is to use the impossibility operator instead of the possibility one. So, given a fuzzy element its approximation is given by the map  $r_i : A \rightarrow A_e \times A_e$  defined as  $r_i(a) := \langle \nu(a), \neg\mu(a) \rangle = \langle \nu(a), \sim a \rangle$ .

We return now to fuzzy and shadowed sets, and we show how it is possible to give them the structure of  $\text{BZMV}^{dM}$  algebras.

**Proposition 2.5** Let  $\mathcal{F} = [0, 1]^X$  be the collection of fuzzy sets on the universe  $X$ . Once defined the operators:

$$\begin{aligned} (f \oplus g)(x) &:= \min\{1, f(x) + g(x)\} \\ \neg f(x) &:= 1 - f(x) \\ \sim f(x) &:= \begin{cases} 1 & \text{if } f(x) = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and the identically zero fuzzy set:  $\mathbf{0}(x) := 0$ ; then, the structure

$\langle \mathcal{F}, \oplus, \neg, \sim, \mathbf{0} \rangle$  is a  $BZMV^{dM}$  algebra.

Similarly, it is possible to give the structure of  $BZMV^{dM}$  algebra to the collection of shadowed sets  $\mathcal{S} = \{0, \frac{1}{2}, 1\}^X$  on the universe  $X$ . The operations syntactically are exactly as in Proposition 2.5, but they are defined on the domain of shadowed sets.

**Proposition 2.6** *Let  $\mathcal{S} = \{0, \frac{1}{2}, 1\}^X$  be the collection of shadowed sets on the universe  $X$ . Then, the structure  $\langle \mathcal{S}, \oplus, \neg, \sim, \mathbf{0} \rangle$ , where  $\oplus, \neg, \sim, \mathbf{0}$  are defined as in proposition 2.5, is a  $BZMV^{dM}$  algebra.*

Now let us consider a fuzzy set  $f \in \mathcal{F}$ . Then, the possibility and necessity of  $f$  are defined, respectively, as

$$\mu(f)(x) = \begin{cases} 0 & f(x) = 0 \\ 1 & f(x) \neq 0 \end{cases} \quad \nu(f)(x) = \begin{cases} 1 & f(x) = 1 \\ 0 & f(x) \neq 1 \end{cases}$$

So, when  $f$  is a fuzzy set, the abstract rough approximation as the pair  $r_i(f) = \langle \nu(f), \sim(f) \rangle$  singles out a shadowed set. In fact,  $\nu(f)$  can be interpreted as the characteristic function of the elements which have value 1 in the induced shadowed set; and  $\sim f$  as the characteristic function of the elements which have value 0. The other elements of the universe  $X$  represents the shadow of the shadowed set.

Precisely, the shadowed set  $s_\alpha$  defined in equation (1) can be obtained, in the case that  $\alpha = 0$ , through a combination of these modal operators:

$$s_0(f) := \mu(f) \odot (\nu(f) \oplus \frac{1}{2}) \quad s_0(f)(x) = \begin{cases} 0 & f(x) = 0 \\ 1 & f(x) = 1 \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad (2)$$

where  $\frac{1}{2}$  is the fuzzy set identically equal to  $\frac{1}{2}$ , i.e., for all  $x \in X$ ,  $\frac{1}{2}(x) := \frac{1}{2}$ .

**Remark 2.7** The mapping  $s_0 : \mathcal{F} \rightarrow \mathcal{S}$ ,  $f \mapsto s_0(f)$ , is not a bijection nor an homomorphism between  $BZMV^{dM}$  algebras, as can be seen in the following counterexample. Let us consider the fuzzy sets  $f_1, f_2 : [0, 1] \mapsto [0, 1]$  defined as:  $f_1(x) := 0.2$  if  $x = 0$  and 0 otherwise, and  $f_2(x) := 0.3$  if  $x = 0$  and 0 otherwise. So,  $f_1 \neq f_2$  but  $s_0(f_1) = s_0(f_2) = 0.5$  if  $x = 0$  and 0 otherwise, and this proves that  $s_0$  is not a bijection. Furthermore (stressing with symbols  $\oplus_S$  and  $\oplus_F$  the “truncated” sum operation acting on  $\mathcal{S}$  and  $\mathcal{F}$  respectively),

$$[s_0(f_1) \oplus_S s_0(f_2)](x) = \begin{cases} 1 & f(x) = 0 \\ 0 & \text{otherwise} \end{cases} \neq \begin{cases} \frac{1}{2} & f(x) = 0 \\ 0 & \text{otherwise} \end{cases} = [s_0(f_1 \oplus_F f_2)](x)$$

and so  $s_0$  is neither an homomorphism of  $BZMV^{dM}$  algebras.

Of course,  $s_0$  gives only the induced shadowed set in the particular case of  $\alpha = 0$ , and it does not capture all the possible ones that can be obtained from

a fuzzy set by equation (1). In order to consider all that possibilities, it is necessary (and sufficient) to generalize the intuitionistic negation as follows:

$$\forall \alpha \in [0, \frac{1}{2}) \quad (\sim_{\alpha} f)(x) := \begin{cases} 1 - f(x) & \text{if } f(x) \leq \alpha \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Clearly, this is a generalization of  $\sim$ , in fact when  $\alpha = 0$  we obtain  $\sim_0 f = \sim f$ . In Figure 2, it is represented a fuzzy set and its  $\alpha$ -impossibility (i.e.,  $\sim_{\alpha}$ ).

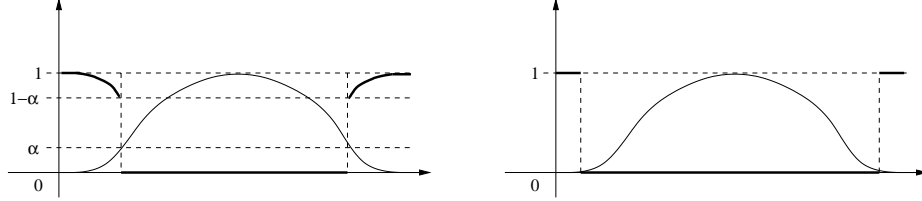


Fig. 2. Generalized  $\sim_{\alpha}$ :  $\alpha \in (0, 1/2)$  and  $\alpha = 0$

The derived operators,  $\mu_{\alpha}$  and  $\nu_{\alpha}$  then become:

$$\begin{aligned} \mu_{\alpha}(f)(x) &:= (\neg \sim_{\alpha} f)(x) = \begin{cases} f(x) & f(x) \leq \alpha \\ 1 & f(x) > \alpha \end{cases} \\ \nu_{\alpha}(f)(x) &:= (\sim_{\alpha} \neg f)(x) = \begin{cases} f(x) & f(x) \geq (1 - \alpha) \\ 0 & f(x) < (1 - \alpha) \end{cases} \end{aligned}$$

Let us introduce the shadowed set  $s_{\alpha}(f)$ , induced by the fuzzy set  $f$  and defined analogously to the (2). This coincide with the shadowed set previously defined by the (1):

$$s_{\alpha}(f) := \mu_{\alpha}(f) \odot (\nu_{\alpha}(f) \oplus \frac{1}{2}) \quad s_{\alpha}(f)(x) = \begin{cases} 0 & f(x) \leq \alpha \\ 1 & f(x) \geq 1 - \alpha \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

So, given a fuzzy set  $f$ , on one side we can obtain the rough approximation  $r_{\alpha}(f) = \langle \nu_{\alpha}(f), \mu_{\alpha}(f) \rangle$ , and on the other side we can induce the shadowed set  $s_{\alpha}(f)$ . The relation between the two functions  $r_{\alpha}$  and  $s_{\alpha}$  is given by the mapping  $r_{\alpha}(f) \rightarrow \psi(r_{\alpha}(f))$  defined as:

$$\psi(r_{\alpha}(f)) = \psi(\langle \nu_{\alpha}(f), \mu_{\alpha}(f) \rangle) = \mu_{\alpha}(f) \odot (\nu_{\alpha}(f) \oplus \frac{1}{2}) = s_{\alpha}(f)$$

In the following diagram all the three functions,  $r_{\alpha}$ ,  $s_{\alpha}$  and  $\psi$  are drawn,

showing the relations among them:

$$\begin{array}{ccc} f & \xrightarrow{r_\alpha} & r_\alpha(f) \\ & s_\alpha \downarrow \psi & \\ & & s_\alpha(f) \end{array}$$

We remark that the function  $\psi$  is not a bijection as can be seen in the following counterexample.

**Example 2.8** Let  $\mathbf{1}$  be the identically one fuzzy set and  $\frac{2}{3}$  the fuzzy set constantly equal to  $\frac{2}{3}$ . We have  $r_{0.4}(\frac{2}{3}) = \langle \frac{2}{3}, \mathbf{1} \rangle$  and  $r_{0.4}(\mathbf{1}) = \langle \mathbf{1}, \mathbf{1} \rangle$ . Then, we obtain  $\psi(r_{0.4}(\frac{2}{3})) = \mathbf{1} = \psi(r_{0.4}(\mathbf{1}))$ .

Coming back to our algebraic structure, we have that, by a substitution of  $\sim$  by  $\sim_\alpha$ , the system  $\langle [0, 1]^X, \oplus, \neg, \sim_\alpha, 0 \rangle$  is no more a  $BZMV^{dM}$  algebra. In fact, for example, axiom  $BZMV6$  is not satisfied. Given a fuzzy set  $f$ , we have:

$$f(x) \oplus (\sim_\alpha \sim_\alpha f)(x) = \begin{cases} 1 & f(x) > \alpha \\ f(x) & f(x) \leq \alpha \end{cases} \neq \begin{cases} 1 & f(x) > \alpha \\ 0 & f(x) \leq \alpha \end{cases} = (\sim_\alpha \sim_\alpha f)(x).$$

Next section is devoted to the study of an algebrization of the structure  $\langle \mathcal{F}, \oplus, \neg, \sim_\alpha, \mathbf{0} \rangle$  containing this new operator  $\sim_\alpha$ .

### 3 pre-BZMV<sup>dM</sup> algebras

We, now, introduce a new algebra, which turns out to be weaker than  $BZMV^{dM}$  algebra. The advantage of this new structure is that it admits as a model the collection of fuzzy sets endowed with the operator  $\sim_\alpha$ .

**Definition 3.1** A structure  $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$  is a pre-BZMV<sup>dM</sup> algebra, if the following are satisfied:

- (i) The substructure  $\langle \mathcal{A}, \oplus, \neg, 0 \rangle$  is a MV algebra, whose induced lattice operators are defined as  $a \vee b := \neg(\neg a \oplus b) \oplus b$ ,  $a \wedge b := \neg(\neg(a \oplus \neg b) \oplus \neg b)$ , and the partial order as  $a \leq b$  iff  $a \wedge b = a$ .
- (ii) The following properties are satisfied:
  - (a)  $a \oplus \sim_w \sim_w a = \neg \sim_w a$
  - (b)  $\sim_w a \leq \neg a$
  - (c)  $\sim_w a \wedge \sim_w b = \sim_w (a \vee b)$
  - (d)  $\sim_w a \vee \sim_w b = \sim_w (a \wedge b)$
  - (e)  $\sim_w \neg a \leq \sim_w \neg \sim_w \neg a$

The collection of all fuzzy sets can be equipped with a structure of pre-BZMV<sup>dM</sup> algebra, according to the following result.

**Proposition 3.2** Let  $\mathcal{F}$  be the collection of fuzzy sets based on the universe  $X$  and let  $\alpha \in [0, \frac{1}{2})$ . Once defined the standard  $\oplus$  and  $\neg$  operators on  $\mathcal{F}$ , and



the  $\sim_\alpha$  negation as in Equation (3), then the structure  $\mathbb{F}_\alpha = \langle \mathcal{F}, \oplus, \neg, \sim_\alpha, \mathbf{0} \rangle$  is a pre-BZMV<sup>dM</sup> algebra, which is not a BZMV<sup>dM</sup> algebra.

In general, it is possible to show that any BZMV<sup>dM</sup> algebra is a pre-BZMV<sup>dM</sup> algebra. In fact, in [3] it is shown that all axioms of definition 3.1 are true in any BZMV<sup>dM</sup> algebra. In general, the vice versa does not hold. For example, let us consider the structure  $\mathbb{F}_\alpha$  with  $\alpha = 0.4$  and  $X = \mathbb{R}$ , and define the fuzzy set  $f(x) = 0.3$  for all  $x \in \mathbb{R}$ . Then,  $\sim_\alpha f(x) \oplus \sim_\alpha \sim_\alpha f(x) = 0.7$  for all  $x$ . So, axiom (BZMV5) is not satisfied.

**Proposition 3.3** *If  $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$  is a pre-BZMV<sup>dM</sup> algebra satisfying  $\sim_w \sim_w a = \neg \sim_w a$ , then it is a BZMV<sup>dM</sup> algebra.*

**Proposition 3.4** *Let  $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$  be a pre-BZMV<sup>dM</sup> algebra. Then, the following properties hold:*

- (i)  $\sim_w 0 = \neg 0$ . In the sequel we set  $1 := \sim_w 0 = \neg 0$ .
- (ii) If  $a \leq b$  then  $\sim_w b \leq \sim_w a$  (contraposition law).

So  $\sim_w$  is a unary operator satisfying both de Morgan laws and the contraposition law (ii). However, it is not an intuitionistic negation, in fact, in general, it satisfies neither the non contradiction law (property B3), nor the weak double negation law (property B1), nor the Brouwer law (i.e., the law  $\forall a, \sim_w a = \sim_w \sim_w \sim_w a$  is not satisfied).

Anyway, also in a pre-BZMV<sup>dM</sup> algebra, it is possible to introduce modal operators of necessity,  $\nu_w(a) := \sim_w \neg a$  and possibility  $\mu_w(a) := \neg \sim_w a$ . However, in this structure  $\nu_w$  and  $\mu_w$  do not have an  $S_5$ -like behavior but only an  $S_4$ -like one (always based on a Kleene lattice instead of on a Boolean one).

**Proposition 3.5** *Let  $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$  be a pre-BZMV<sup>dM</sup> algebra. Then, for every  $a \in \mathcal{A}$  the following properties are satisfied:*

- (i)  $\nu_w(a) \leq a \leq \mu_w(a)$  [T principle]
- (ii)  $\nu_w(\nu_w(a)) = \nu_w(a)$   $\mu_w(\mu_w(a)) = \mu_w(a)$  [ $S_4$  principle]

In general the properties:  $a \leq \nu_w(\mu_w(a))$ ,  $\mu_w(a) = \nu_w(\mu_w(a))$ , and  $\nu_w(a) = \mu_w(\nu_w(a))$  do not hold. As an example, let us consider the algebra  $\mathbb{F}_{0.4}$ , with  $X = [0, 1]$ , and define the fuzzy set  $f(x) = 0.3$  if  $x < \frac{1}{2}$  and  $f(x) = 0.7$  otherwise. We have:

$$\begin{aligned} \mu_\alpha(f(x)) &= \begin{cases} 0.3 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} \neq \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} = \nu_\alpha(\mu_\alpha(f(x))). \\ \nu_\alpha(f(x)) &= \begin{cases} 0 & x < \frac{1}{2} \\ 0.7 & x \geq \frac{1}{2} \end{cases} \neq \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} = \mu_\alpha(\nu_\alpha(f(x))). \end{aligned}$$

Finally,  $f(x)$  is incomparable with  $\nu_\alpha(\mu_\alpha(f(x)))$ .

Even if the necessity and possibility mappings have a weaker modal behavior in  $\text{pre-BZVM}^{dM}$  algebras than in  $\text{BZMV}^{dM}$  algebras, they can still be used to define a lower and upper approximation, and it turns out that  $\nu_w$  is an interior operator and  $\mu_w$  is a closure operator.

**Proposition 3.6** *Let  $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$  be a pre-BZMV $^{dM}$  algebra. Then the map  $\mu_w : \mathcal{A} \rightarrow \mathcal{A}$  such that  $\mu_w(a) := \neg \sim_w a$  is a closure operator. That is, the following are satisfied:*

$$\begin{array}{lll} (C_0) & 0 = \mu_w(0) & (\text{normalized}) \\ (C_1) & a \leq \mu_w(a) & (\text{increasing}) \\ (C_2) & \mu_w(a) = \mu_w(\mu_w(a)) & (\text{idempotent}) \\ (C_3) & a \leq b \text{ implies } \mu_w(a) \leq \mu_w(b) & (\text{monotone}) \end{array}$$

The collection of all *closed* sets is then defined as

$$\mathbb{C}(\mathcal{A}) = \{a \in \mathcal{A} : a = \mu_w(a)\}$$

**Proposition 3.7** *Let  $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$  be a pre-BZMV $^{dM}$  algebra. Then the map  $\nu_w : \mathcal{A} \rightarrow \mathcal{A}$  such that  $\nu_w(a) := \sim_w \neg a$  is an interior operator, i.e.:*

$$\begin{array}{lll} (I_0) & 1 = \nu_w(1) & (\text{normalized}) \\ (I_1) & \nu_w(a) \leq a & (\text{decreasing}) \\ (I_2) & \nu_w(a) = \nu_w(\nu_w(a)) & (\text{idempotent}) \\ (I_3) & a \leq b \text{ implies } \nu_w(a) \leq \nu_w(b) & (\text{monotone}) \end{array}$$

The collection of all *open* sets is then defined as

$$\mathbb{O}(\mathcal{A}) = \{a \in \mathcal{A} : a = \nu_w(a)\}.$$

It is possible to show that, in general, these subsets of  $\mathcal{A}$  do not coincide, neither one is a subset of the other. So, it is worthwhile to consider also the set of all *clopen* elements, i.e. elements which are both closed and open:  $\mathbb{CO}(\mathcal{A}) = \mathbb{C}(\mathcal{A}) \cap \mathbb{O}(\mathcal{A})$ .

The above considerations lead to the definition of an abstract approximation space generated by a pre-BZMV $^{dM}$  algebra.

**Definition 3.8** Let  $\mathbf{A}$  be a pre-BZMV $^{dM}$  algebra. The induced rough approximation space is the structure  $\langle \mathcal{A}, \mathbb{O}(\mathcal{A}), \mathbb{C}(\mathcal{A}), \nu_w, \mu_w \rangle$ , where  $\mathcal{A}$  is the set of approximable elements;  $\mathbb{O}(\mathcal{A}) \subseteq \mathcal{A}$  is the set of *innerdefinable* elements, such that  $0$  and  $1 \in \mathbb{O}(\mathcal{A})$ ;  $\mathbb{C}(\mathcal{A}) \subseteq \mathcal{A}$  is the set of *outerdefinable* elements, such that  $0$  and  $1 \in \mathbb{C}(\mathcal{A})$ ;  $\nu_w : \mathcal{A} \rightarrow \mathbb{O}(\mathcal{A})$  is the *inner approximation map*;

$\mu_w : \mathcal{A} \rightarrow \mathbb{C}(\mathcal{A})$  is the *outer approximation map*. For any element  $a \in \mathcal{A}$ , its *rough approximation* is defined as the pair:

$$r_w(a) := \langle \nu_w(a), \mu_w(a) \rangle \quad [\text{with } \nu_w(a) \leq a \leq \mu_w(a)]$$

drawn in the following diagram:

$$\begin{array}{ccc} & a \in \mathcal{A} & \\ \nu_w \swarrow & \downarrow r_w & \searrow \mu_w \\ \nu_w(a) \in \mathbb{O}(\mathcal{A}) & & \mu_w(a) \in \mathbb{C}(\mathcal{A}) \\ & \langle \nu_w(a), \mu_w(a) \rangle & \end{array}$$

This approximation is the best approximation by open (resp. closed) elements that it is possible to define on a pre-BZMV<sup>dM</sup> structure, i.e., there hold properties similar to (I1)–(I3) and (O1)–(O3), the only difference is that here we have to distinguish between open-exact and closed-exact elements.

In the context of the fuzzy sets pre-BZMV<sup>dM</sup> algebra of proposition 3.2 the collection of open and closed elements are respectively:

$$\begin{aligned} \mathbb{C}(\mathcal{F}) &= \{f \in \mathcal{F} : f(x) > \alpha \text{ iff } f(x) = 1\} \\ \mathbb{O}(\mathcal{F}) &= \{f \in \mathcal{F} : f(x) < 1 - \alpha \text{ iff } f(x) = 0\} \end{aligned}$$

The clopen sets are the 0–1 valued fuzzy sets,  $\mathbb{CO}(\mathcal{F}) = \{0, 1\}^X$ . In the universe  $[0, 1]$ , once set  $\alpha = 0.4$ , an example of open element is  $f_1(x) = 0$  if  $x < \frac{1}{2}$  and 0.7 otherwise and an example of closed set is  $f_2(x) = 0.3$  if  $x < \frac{1}{2}$  and 1 otherwise. The fuzzy sets  $f_1$  and  $f_2$  are drawn in Figure 3.

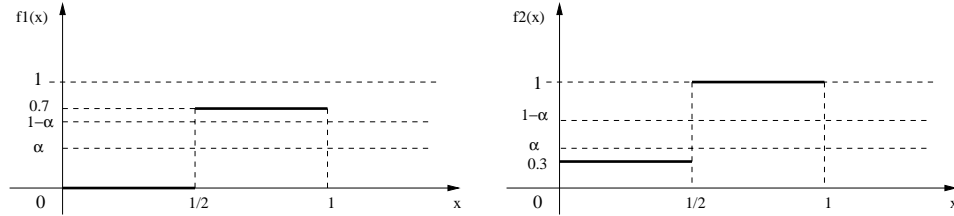


Fig. 3. Example of open fuzzy set,  $f_1$ , and closed fuzzy set,  $f_2$ .

Finally, we remark that we can also enrich the collection of shadowed sets  $\{0, \frac{1}{2}, 1\}^X$  with the operation  $\sim_\alpha$ , in order to give it a pre-BZMV<sup>dM</sup> structure. However, it can be easily proved that in this case  $\sim_\alpha$  is equivalent to  $\sim_0$  for all  $\alpha \in [0, \frac{1}{2})$  and so we again obtain a BZMV<sup>dM</sup> algebra.

## 4 Conclusions

In this paper we analyzed shadowed sets from the algebraic point of view. As a first result we have seen that once properly defined the operators on the

collection of shadowed sets of a given universe, they result to be a  $BZMV^{dM}$  algebra. The same can be proved about the collection of fuzzy sets. Moreover, in such a structure it is possible to algebraically define an operator which given a fuzzy set returns a particular induced shadowed set. In order to generalize such an operator it was necessary to introduce the new structure of pre- $BZMV^{dM}$  algebras. Finally, it was shown that the collection of fuzzy sets with a generalized notion of intuitionistic negation is a model of pre- $BZMV^{dM}$  algebras. A possible development of the present work is a deeper theoretical analysis of pre- $BZMV^{dM}$  algebras, which involves the study of the independence of its axioms, the proof of a representation and a completeness theorem. On the other hand, it would also be interesting to analyze the implications of such a structure in an application context.

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